



2024-2025
Fall Semester



Course of Power System Analysis

Derivation of transmission lines equations from electromagnetic field theory

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The modeling of electrical transmission lines (for both transmission and distribution of electricity) represents a fundamental element to perform the following analysis:

- **evaluation of the voltage profile along the line;**
- **balance of reactive powers;**
- **losses evaluation for the assessment of the electricity transportation efficiency;**
- **definition of the line models for load flow calculation;**
- **modeling of ampacity and congestion constrains;**
- **...**

Outline

Hypotheses

From the field theory to
transmission line equivalent
circuit

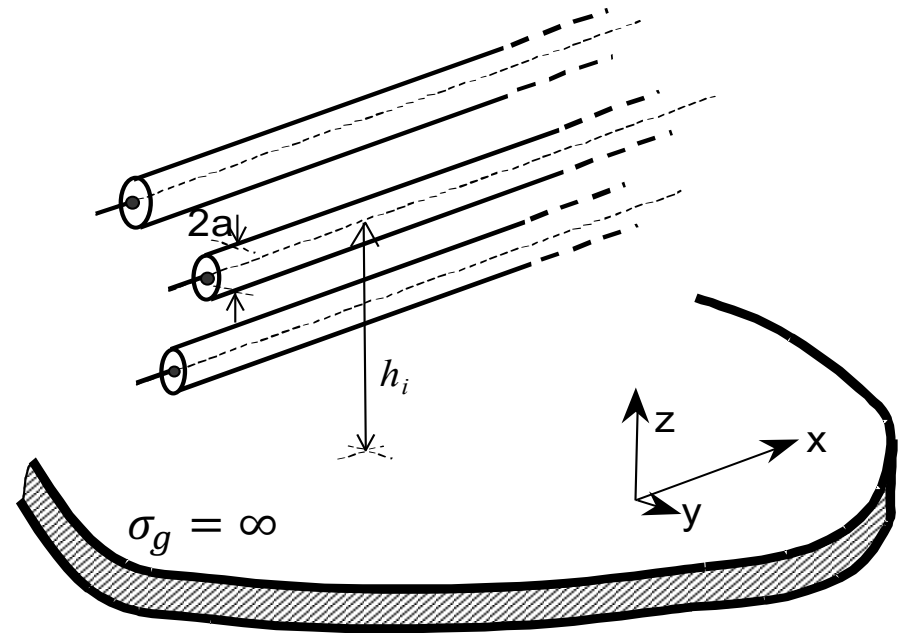
The line distributed parameters
equivalent circuit

Hypotheses

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Let us refer to the geometry in the figure.

1. The line geometry is **uniform**;
2. The line is **infinitely long**;
3. The line is composed by n conductors, parallel among each other and with respect to the ground. The height with respect to the ground of the i^{th} conductor is h_i . The conductors are made of an electrically conductive material of finite conductivity σ_c . The ground is assumed to have an infinite conductivity, $\sigma_g = \infty$.

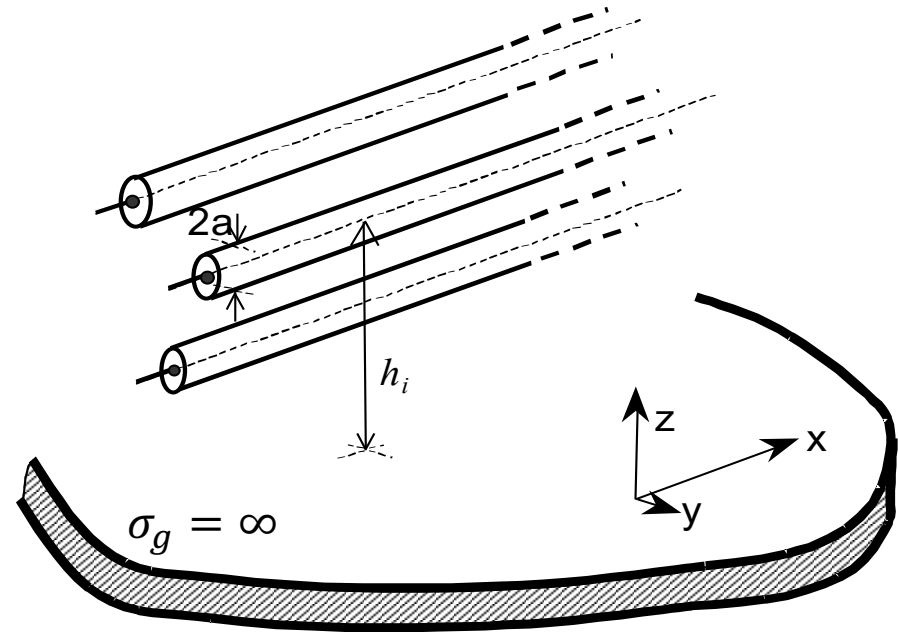


General conductors' geometry of an electrical transmission line.

Hypotheses

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4. The **transverse dimensions (cross sectional dimensions)** of the line are small compared to the minimum wavelength λ_{min} of the electrical propagation phenomena taking place along the **line**. We can then consider that **propagation occurring only along the x axis**, and, as we shall see, the line can be represented by a distributed-parameter structure along its x -axis.



General conductors' geometry of an electrical transmission line.

Observation: is HP4 verified in practice ?

We remind that the wavelength λ of the propagative phenomena taking place along the line is associated to the frequency f of the wavelength and its propagation speed v :

$$v = \lambda f$$

Let us consider an overhead line, in this case the value of v is very close to the speed of light, namely $2.9979 \cdot 10^8 \frac{m}{s}$, therefore assuming the European nominal frequency of $f = 50 \text{ Hz}$, we get:

$$\lambda = \frac{2.9979 \cdot 10^8 \frac{m}{s}}{50 \text{ Hz}} \cong 6 \cdot 10^6 m.$$

In view of the above, **we can conclude that Hp4 is satisfied in practice. It is interesting to note that Hp4 is satisfied also for larger frequencies. Therefore, the model we are about to obtain can be applied to study phenomena with higher frequencies (i.e., up to some MHz).**

5. The **line response is quasi-TEM (transverse electromagnetic), i.e. the electromagnetic field outside the conductors produced by the electric charges and currents along them, is confined in the transverse plane perpendicular to the line x -axis.** (Note that it is practically impossible to have the response of a line to be purely TEM. In fact, a pure TEM mode would occur only for the case of a perfectly conducting conductor and ground).
6. The **sum of the line currents at any cross section of the line is zero,** i.e. the ground – *the reference conductor* – is the return path for the currents in the n overhead conductors. (This means that we are considering only ‘transmission line mode’ currents and neglecting the so-called ‘antenna-mode’ currents. If we desire to compute the load responses of the line, this assumption is adequate, because the antenna mode current response is small near the ends of the line. Along the line, however, the presence of antenna-mode currents makes that the sum of the currents at a cross section is not necessarily equal to zero. However, the quasi-symmetry due to the presence of the ground plane results in a very small contribution of antenna mode currents and consequently, the predominant mode on the line will be transmission line.)

Outline

Hypotheses

From the field theory to
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From field theory to equivalent circuit

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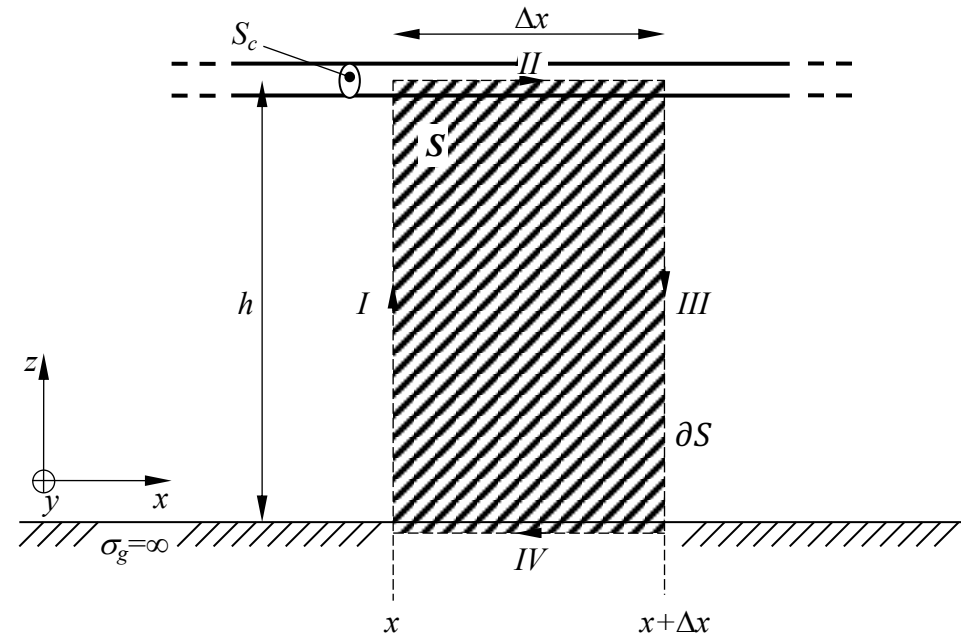
To simplify the derivation, we will refer to a single-conductor line (see figure)

Let us recall the first Maxwell's equation written in the **time** and **frequency domains**, respectively:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

where $\mathbf{E}$, $\mathbf{B}$ are the vectors of the electric and magnetic fields. Let us apply the **Stoke's theorem** to the previous equation in the **frequency-domain** and to the **oriented surface S with boundary ∂S** .

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = -j\omega \iint_S \mathbf{B} \cdot d\mathbf{S}$$



Representation of a section Δx of a single-conductor overhead transmission line.

Note: in the coming slides there is an abuse of nomenclature since we will indicate the electric field with the letter E that, so far, has been used to indicate the phase-to-ground voltage.

From field theory to equivalent circuit

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We can write the first term of the previous equation as the sum of the integrals over the edges of the boundary ∂S .

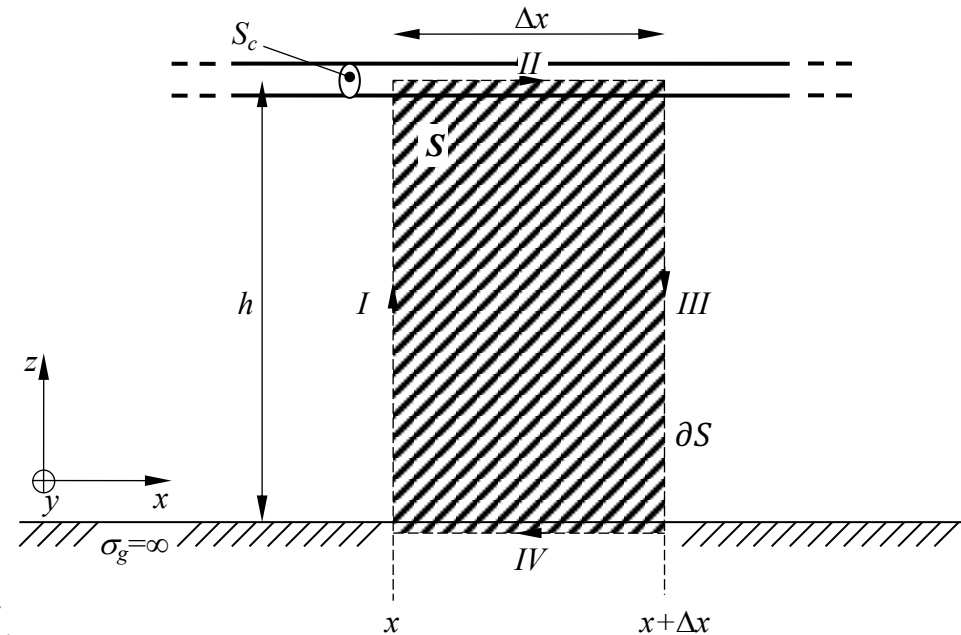
Due to Hp4 and Hp5, we have that the **integration of the electric field along the edges I and III** in the figure, **contains only the vertical component of the electric field (E_z)**.

Furthermore, thanks to Hp3 ($\sigma_g = \infty$), we have that the **electric field inside the ground is zero, so it is its integral along the edge IV**.

So we get:

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = - \int_0^h (E_z(x + \Delta x, 0, z) - E_z(x, 0, z)) dz + \int_x^{x+\Delta x} E_x(x, 0, h) dx$$

It is worth noting that **Hp5 applies outside the conductors**. Therefore, the integral of the electric field on edge II is $\neq 0$ (i.e., field along the conductor) and **corresponds to the conductor's voltage over Δx** .



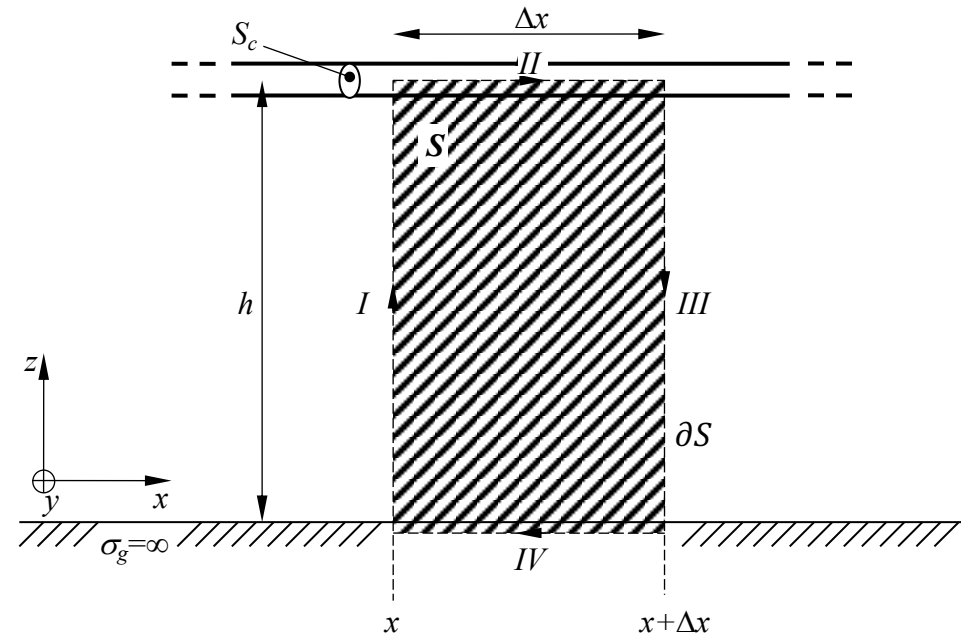
Representation of a section Δx of a single-conductor overhead transmission line.

From field theory to equivalent circuit

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Regarding the magnetic field, Hp5 allows to conclude that, if in the conductor there is a current I , the produced magnetic field is perpendicular to the surface S and can be represented by the sole component B_y . Therefore, we can write the

$$\begin{aligned} -j\omega \iint_S \mathbf{B} \cdot d\mathbf{S} &= \\ &= -j\omega \int_x^{x+\Delta x} \int_0^h B_y(x, 0, z) dz dx \end{aligned}$$



Representation of a section Δx of a single-conductor overhead transmission line.

From field theory to equivalent circuit

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From the previous equations, we can derive the following three limits:

$$\lim_{\Delta x \rightarrow 0} -\frac{1}{\Delta x} \int_0^h (E_z(x + \Delta x, 0, z) - E_z(x, 0, z)) dz = -\frac{\partial}{\partial x} \int_0^h E_z(x, 0, z) dz$$

$$\lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} E_x(x, 0, h) dx = E_x(x, 0, h)$$

$$\lim_{\Delta x \rightarrow 0} -\frac{1}{\Delta x} j\omega \int_x^{x+\Delta x} \int_0^h B_y(x, 0, z) dz dx = -j\omega \int_0^h B_y(x, 0, z) dz$$

In what follows, we will **simplify these three limits to help the derivation of an equivalent electrical circuit of the line.**

From field theory to equivalent circuit

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The very first integral quantity to introduce **derives from Hp3–Hp5**. Indeed, thanks to these Hps, we have that the **electric field produced by the line's conductor, is conservative**. Therefore, we can define the phase-to-ground voltage as follows (note that is **path independent**):

$$\bar{U}(x) = - \int_0^h E_z(x, 0, z) dz$$

From the definition of the phase-to-ground voltage, we have that the first of the three limits can be written as follows:

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} - \frac{1}{\Delta x} \int_0^h (E_z(x + \Delta x, 0, z) - E_z(x, 0, z)) dz &= - \frac{\partial}{\partial x} \int_0^h E_z(x, 0, z) dz = \\ &= \frac{d\bar{U}(x)}{dx} \end{aligned}$$

Where $\bar{U}(x)$ is the phasor of the conductor's phase-to-ground voltage in correspondence of a generic point x of the line.

From field theory to equivalent circuit

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The result of the second limit

$$\lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} E_x(x, 0, h) dx = E_x(x, 0, h)$$

can be written in a different way by reminding the **link between the electric field and the current density inside a conductor** $J = \sigma_c E$. In particular, by **assuming the skin effect being negligible**, we have that **the current density J is constant over the conductor's cross section S_c** . Therefore, we have:

$$\bar{J}(x) = \frac{\bar{I}(x)}{S_c} = \sigma_c E_x(x, 0, h) \rightarrow E_x(x, 0, h) = \frac{1}{S_c \sigma_c} \bar{I}(x) = \rho_c \frac{1}{S_c} \bar{I}(x)$$

Where $\bar{I}(x)$ is the **phasor of the current circulating on the conductor in a generic point x of the line** and σ_c, ρ_c the conductor's electrical conductivity and resistivity, respectively. From the previous equation, we can observe that the term $\rho_c \frac{1}{S_c}$ is the **per-unit of length conductor's electrical resistance r** . So, we have:

$$E_x(x, 0, h) = r \bar{I}(x)$$

From field theory to equivalent circuit

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The result of the third limit

$$\lim_{\Delta x \rightarrow 0} -\frac{1}{\Delta x} j\omega \int_x^{x+\Delta x} \int_0^h B_y(x, 0, z) dz dx = -j\omega \int_0^h B_y(x, 0, z) dz$$

can be written in a different way by leveraging Hp4 and Hp5. Indeed, we can say that the magnetic field $B_y(x, 0, z)$ is related to the current circulating along the conductor $\bar{I}(x)$ by means of the conductor's **per-unit of length induction coefficient l'** :

$$l' = \frac{\int_0^h B_y(x, 0, z) dz}{\bar{I}(x)}$$

From field theory to equivalent circuit

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Thanks to these developments, we can rewrite

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = -j\omega \iint_S \mathbf{B} \cdot d\mathbf{S}$$
$$= - \int_0^h (E_z(x + \Delta x, 0, z) - E_z(x, 0, z)) dz + \int_x^{x+\Delta x} E_x(x, 0, h) dx$$

as follows:

$$\frac{d\bar{U}(x)}{dx} + r\bar{I}(x) + j\omega l'\bar{I}(x) = 0$$

or, by defining the **per-unit length impedance of the line** $\bar{z} = r + j\omega l'$

$$\frac{d\bar{U}(x)}{dx} + \bar{z}\bar{I}(x) = 0$$

The above is the **first transmission lines (or telegraphers') equation**.

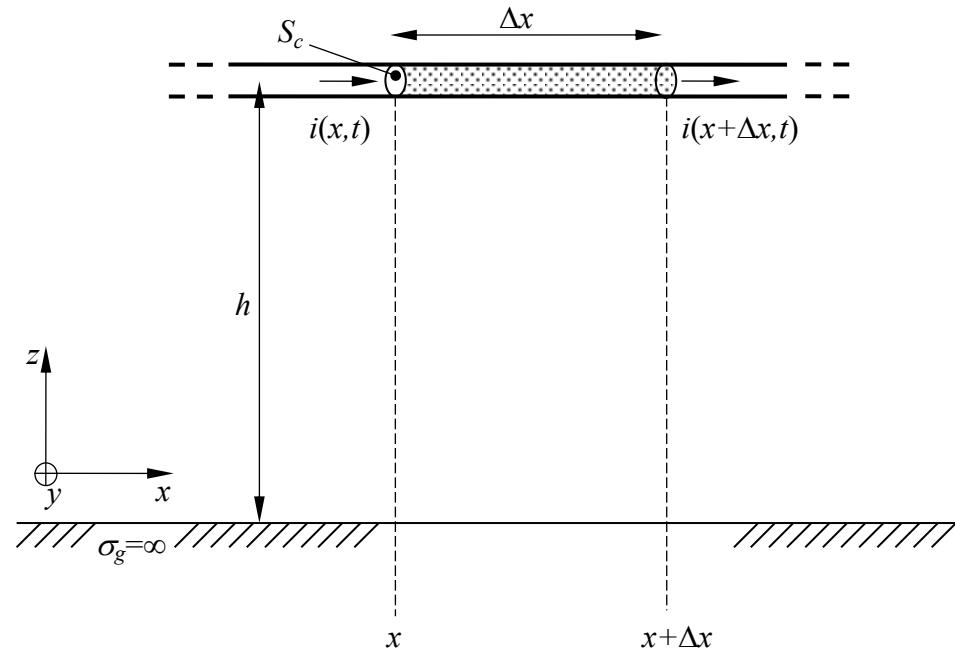
From field theory to equivalent circuit

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The second **transmission lines equation**, can be derived by applying the **charge conservation law** to the conductor's cylindrical section of the line in the figure.

By indicating the **conductor's charge density per-unit of length** as $q'(x, t)$, we can apply the charge conservation law to the conductor's cylinder of section S_c and length Δx :

$$\begin{aligned} i(x, t) - i(x + \Delta x, t) \\ = \frac{d}{dt} (q'(x, t) \Delta x) \end{aligned}$$



By re-writing the above equation in the **frequency domain**, we get:

$$\frac{\bar{I}(x) - \bar{I}(x + \Delta x)}{\Delta x} = j\omega \bar{q}'(x) \xrightarrow{\Delta x \rightarrow 0} \frac{d\bar{I}(x)}{dx} = -j\omega \bar{q}'(x)$$

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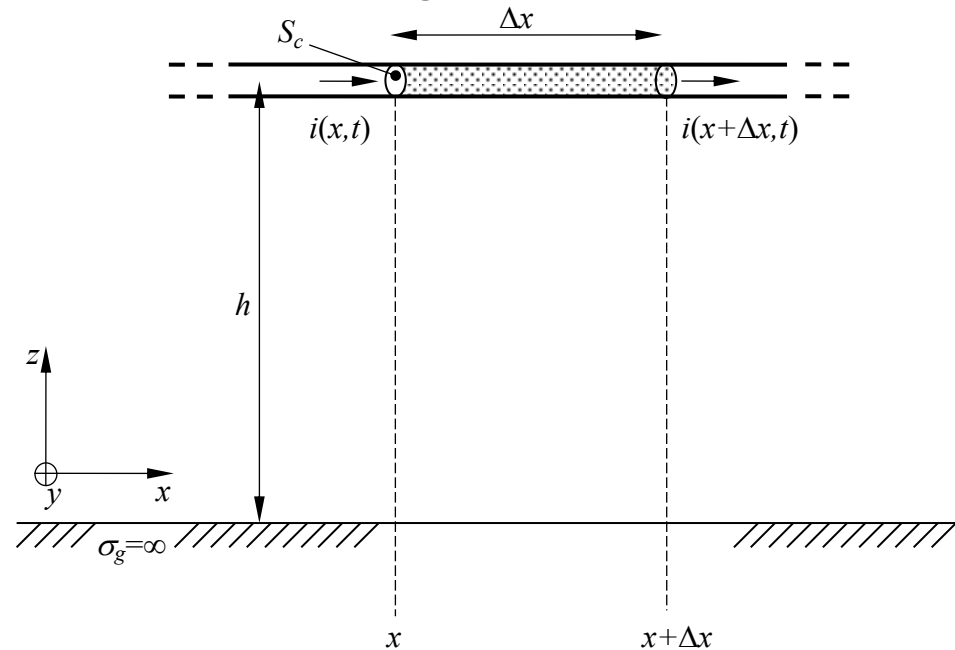
The term $\bar{q}(x)$ can be re-written by considering that the line conductor and the ground are an electrical structure that can be assimilated to a capacitor. Indeed, the line conductor and the ground are **two**

conductive elements separated by a dielectric material (i.e. air). Therefore, the conductor's charge density per-unit of length $\bar{q}'(x)$ can be linked to the conductor's voltage $\bar{U}(x)$ via the line per-unit length capacitance c'

$$\bar{q}'(x) = c'\bar{U}(x)$$

Therefore, we have:

$$\frac{d\bar{I}(x)}{dx} + j\omega c'\bar{U}(x) = 0$$



From field theory to equivalent circuit

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In case we would like to **take into account the fact that the electrical insulator between the conductor's line and the ground is not ideal** (i.e., it has active power losses), we need to admit the presence of a **resistive element in parallel to the capacitance c' in the form of a per-unit length conductance g .**

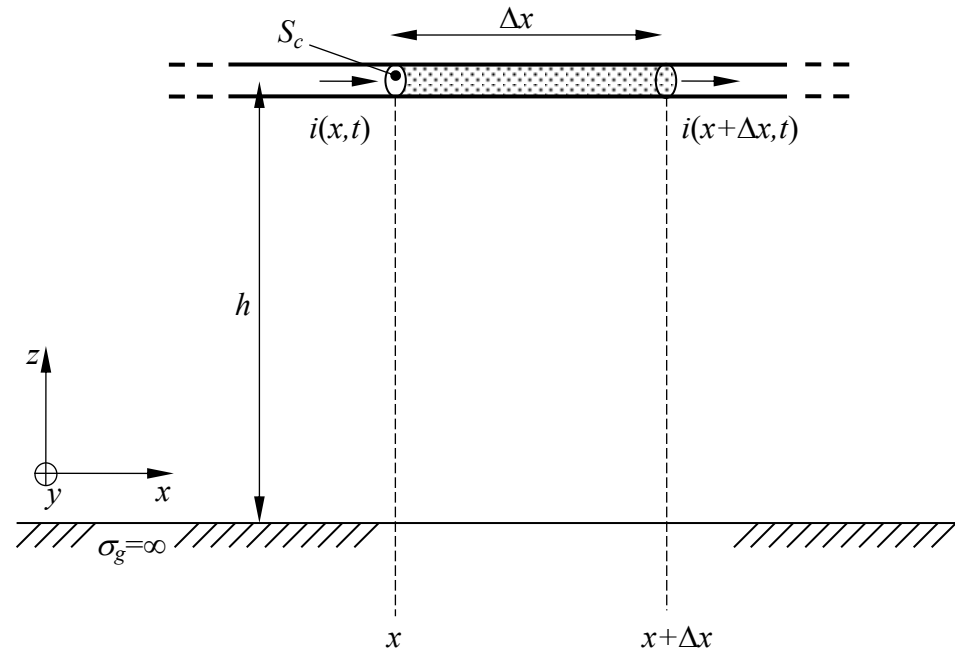
In this case, the previous equation becomes:

$$\frac{d\bar{I}(x)}{dx} + g\bar{U}(x) + j\omega c'\bar{U}(x) = 0$$

or, by defining the **per-unit length shunt admittance of the line $\bar{y} = g + j\omega c'$** , we have:

$$\frac{d\bar{I}(x)}{dx} + \bar{y}\bar{U}(x) = 0$$

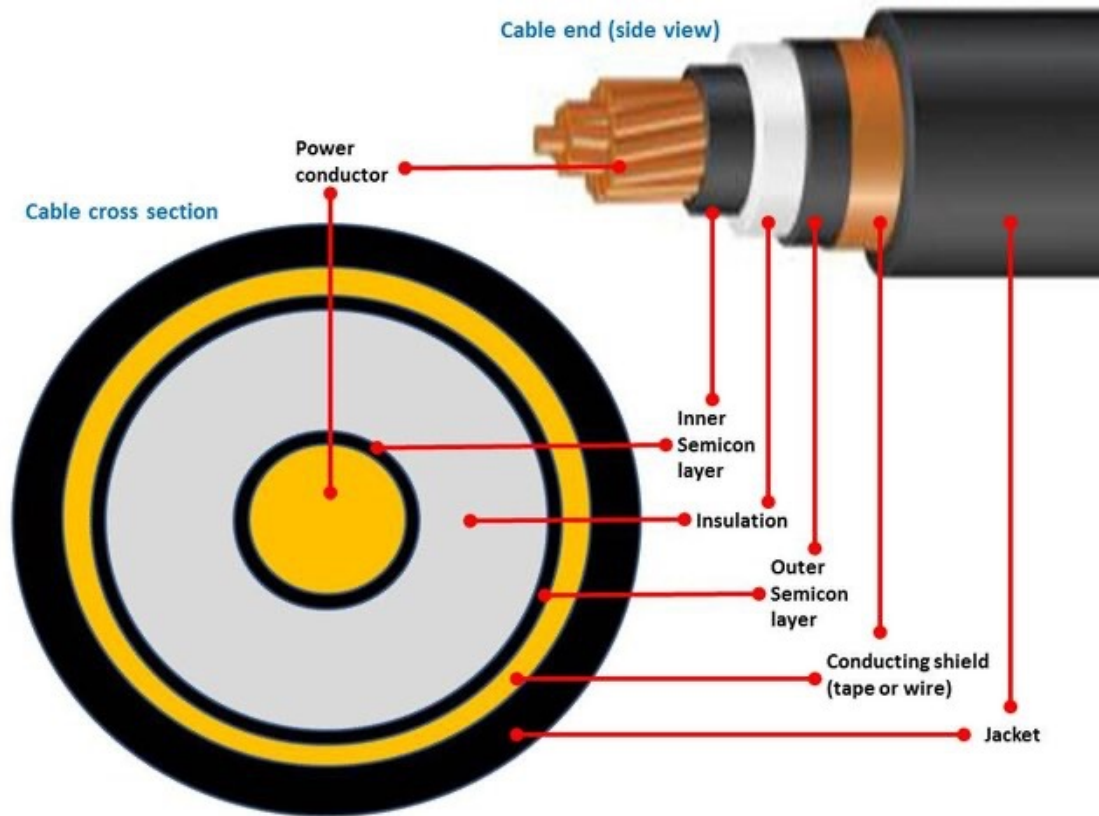
The above is the **second transmission lines (or telegraphers') equation**.



From field theory to equivalent circuit

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It is worth noting that the same derivation can be obtained if we consider a transmission line geometry given by a **coaxial cable**.



Outline

Hypotheses

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The line distributed parameters
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The distributed parameters circuit

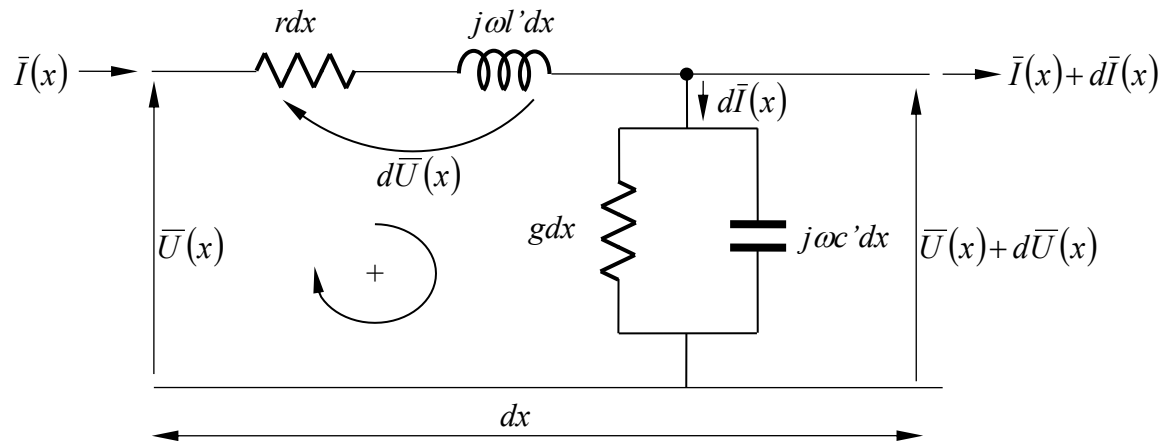
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It is interesting to observe that the two transmission lines (or telegraphers') equations:

$$\frac{d\bar{U}(x)}{dx} + \bar{z}\bar{I}(x) = 0$$

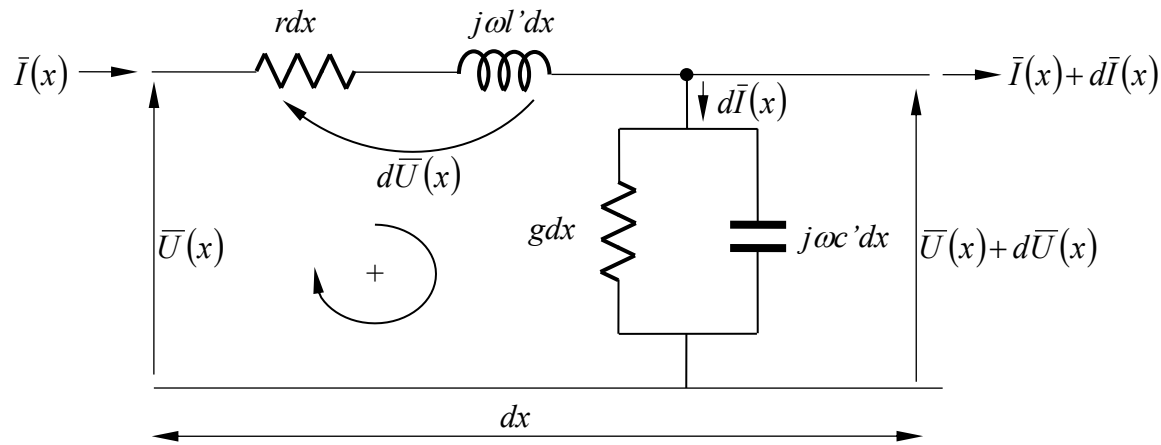
$$\frac{d\bar{I}(x)}{dx} + \bar{y}\bar{U}(x) = 0$$

are equivalent to the following electrical circuit associated to an infinitesimal length dx of the line.



The distributed parameters circuit

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Indeed, the first transmission line equation can be obtained by **applying the KVL to the loop indicated in the figure:**

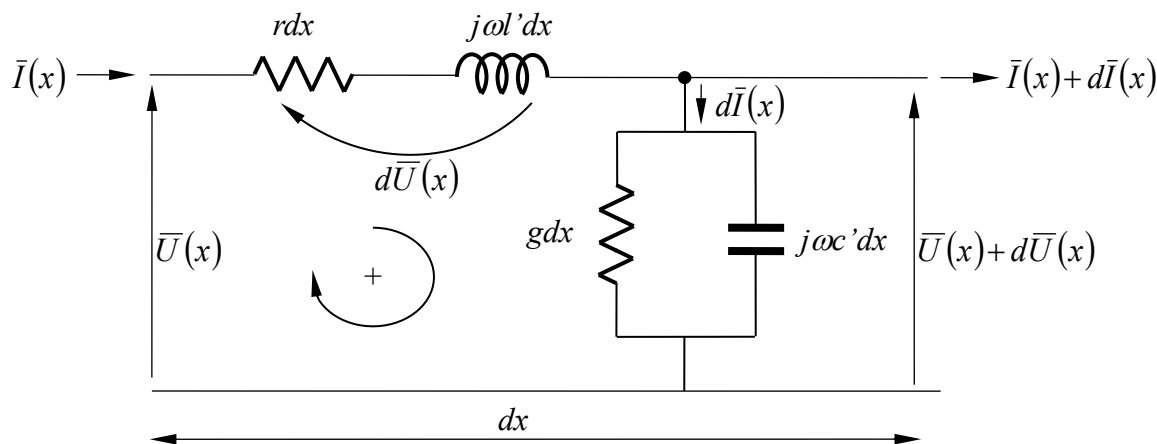
$$\bar{U}(x) = (r + j\omega l')dx\bar{I}(x) + \bar{U}(x) + d\bar{U}(x)$$

$$d\bar{U}(x) + (r + j\omega l')dx\bar{I}(x) = 0$$

$$\frac{d\bar{U}(x)}{dx} + \bar{z}\bar{I}(x) = 0$$

The distributed parameters circuit

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While the second transmission line equation can be obtained by **applying the KCL to the node indicated in the figure**:

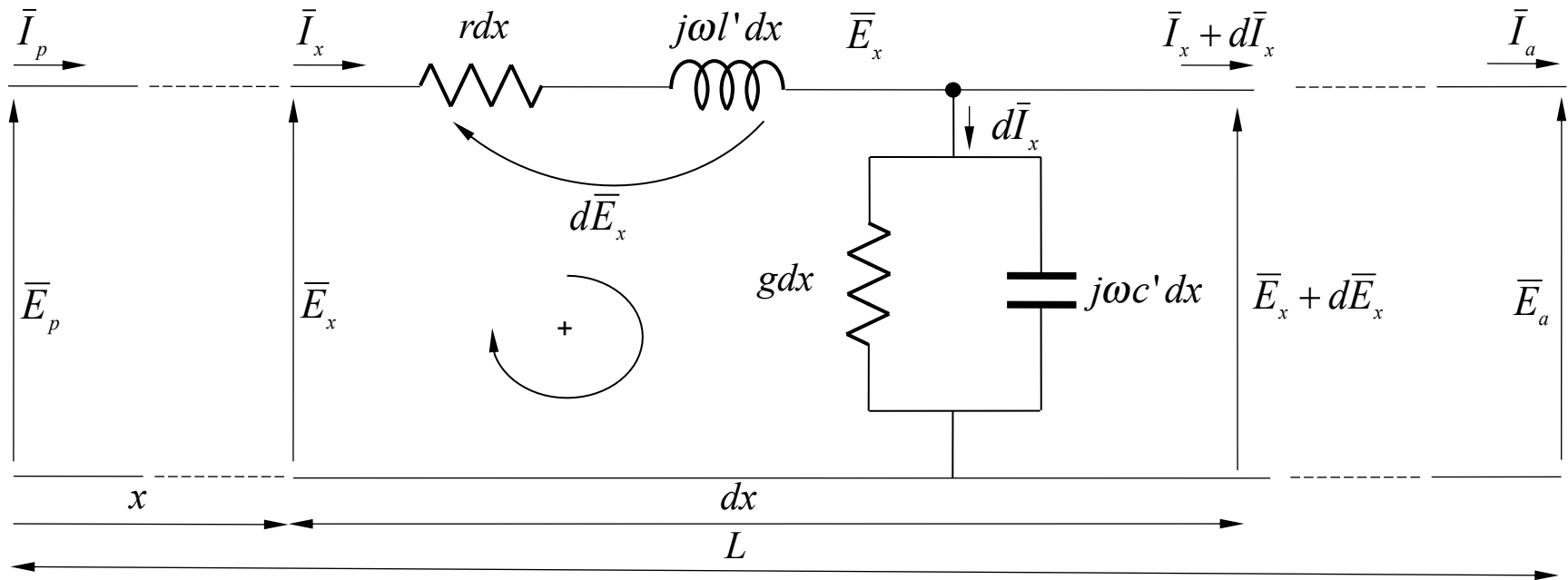
$$\bar{I}(x) = (g + j\omega c')dx(\bar{U}(x) + d\bar{U}(x)) + \bar{I}(x) + d\bar{I}(x)$$

$$d\bar{I}(x) + (g + j\omega c')dx\bar{U}(x) = 0$$

$$\frac{d\bar{I}(x)}{dx} + \bar{y}\bar{U}(x) = 0$$

Note: in the above derivation, the second-order differential $d\bar{U}(x)dx$ has been neglected with respect to the first order one $d\bar{I}(x)$.

The equivalent electrical circuit associated to an infinitesimal length dx of the line **can be extended along the whole length of the line and obtain the so-called distributed parameters equivalent circuit (or model).**



Note: in the above figure there is a **change of nomenclature regarding the phase-to-ground voltage**: indeed, to be coherent with the symbols used in three-phase circuits, this voltage is indicated as $\bar{E}$.